

plus p times the number of those d_i which are equal or greater than p . (In other words, $N(p)$ is the area of the part of Young diagram $(d_1 + 1, \dots, d_n + 1)$ strictly to the left from the $(p + 1)$ th column.) Let the index $\Upsilon(p)$ be equal to the number of even elements $d_i \geq p$ if p is even, and to the number of odd elements $d_i \geq p$ if p is odd. By $\tilde{H}^*(X)$ we denote the cohomology group reduced modulo a point. $\overline{H}_*(X)$ denotes the Borel–Moore homology group, i.e. the homology group of the one-point compactification of X reduced modulo the added point.

Theorem 1. *If the space $\mathbb{R}^D \setminus \Sigma$ is non-empty (i.e. either $n > 1$ or d_1 is even) then the group $\tilde{H}^*(\mathbb{R}^D \setminus \Sigma, \mathbb{Z})$ is equal to the direct sum of following groups:*

A) *For any $p = 1, \dots, d_3$,*

if $\Upsilon(p)$ is even, then \mathbb{Z} in dimension $N(p) - 2p$ and \mathbb{Z} in dimension $N(p) - 2p + 1$,

if $\Upsilon(p)$ is odd, then only one group \mathbb{Z}_2 in dimension $N(p) - 2p + 1$;
B) *If $d_1 - d_2$ is odd then only one additional summand \mathbb{Z} in dimension $D - d_1 - d_2 - 2$. If $d_1 - d_2$ is even, then additional summand $\mathbb{Z}^{d_2 - d_3 + 1}$ in dimension $D - d_1 - d_2 - 1$ and (if $d_2 \neq d_3$) summand $\mathbb{Z}^{d_2 - d_3}$ in dimension $D - d_1 - d_2 - 2$.*

Example 1. Let $n = 2$. If d_1 and d_2 are of the same parity, then $\mathbb{R}^D \setminus \Sigma$ consists of $d_2 + 1$ connected components, each of which has the homology of a circle. For the invariant separating systems from different components we can take the index of the induced map of the unit circle $S^1 \subset \mathbb{R}^2$ into $\mathbb{R}^2 \setminus 0$. This index can take all values of the same parity as d_2 from the segment $[-d_2, d_2]$. The 1-dimensional cohomology class inside any component is just the rotation index of the image of a fixed point (say, $(1, 0)$) around the origin.

If d_1 and d_2 are of different parities, then the space $\mathbb{R}^D \setminus \Sigma$ has the homology of a two-point set. The index separating its two connected components can be calculated as the parity of the number of zeros of the odd-degree polynomial of our non-resultant system, which lie in the (well-defined) domain in \mathbb{RP}^1 where the even-degree polynomial is positive.

Now, let \mathbb{C}^D be the space of all polynomial systems (1) with complex coefficients $a_{i,j}$, and $\Sigma_{\mathbb{C}} \subset \mathbb{C}^D$ the set of systems having solutions in $\mathbb{C}^2 \setminus 0$.

Theorem 2. *For any $n > 1$ the group $H^*(\mathbb{C}^D \setminus \Sigma_{\mathbb{C}}, \mathbb{Q})$ is isomorphic to \mathbb{Q} in dimensions $0, 2n - 3, 2n - 1$ and $4n - 4$, and is trivial in all other dimensions.*

Consider also the space \mathbb{C}^{d+1} of all complex homogeneous polynomials

$$a_0x^d + a_1x^{d-1}y + \cdots + a_dy^d$$

and m -discriminant Σ_m in it, consisting of all polynomials vanishing on some line with multiplicity $\geq m$.

Theorem 3. *For any $m > 1$ and $d \geq 2m$, the group $H^*(\mathbb{C}^{d+1} \setminus \Sigma_m, \mathbb{Q})$ is isomorphic to \mathbb{Q} in dimensions $0, 2m-3, 2m-1$ and $4m-4$, and is trivial in all other dimensions. For any $m > 1$ and $d \in [m, 2m-1]$ this group is isomorphic to \mathbb{Q} in dimensions $0, 2m-3, 2m-1$ and $2d-2$, and is trivial in all other dimensions.*

3. SOME PRELIMINARY FACTS

Denote by $B(M, p)$ the *configuration space* of subsets of cardinality p in the topological space M .

Lemma 1. *For any natural p , there is a locally trivial fibre bundle $B(S^1, p) \rightarrow S^1$, whose fiber is homeomorphic to \mathbb{R}^{p-1} . This fibre bundle is non-orientable if p is even, and is orientable (and hence trivial) if p is odd. \square*

Indeed, the projection of this fibre bundle can be realised as the product of p points of the unit circle in \mathbb{C}^1 . The fibre of this bundle can be identified in the terms of the universal covering $\mathbb{R}^p \rightarrow T^p$ with any connected component of a hyperplane $\{x_1 + \cdots + x_p = \text{const}\}$ from which all affine planes given by $x_i = x_j + 2\pi k$, $i \neq j$, $k \in \mathbb{Z}$, are removed. Such a component is convex and hence diffeomorphic to \mathbb{R}^{p-1} . The assertion on orientability can be checked immediately. \square

Let us embed a manifold M generically into the space \mathbb{R}^T of a very large dimension, and denote by M^{*r} the union of all $(r-1)$ -dimensional simplices in \mathbb{R}^T , whose vertices lie in this embedded manifold (and the “genericity” of the embedding means that if two such simplices have a common point in \mathbb{R}^T , then their minimal faces, containing this point, coincide). \square

Proposition 1 (C. Caratheodory theorem, see also [6]). *For any $r \geq 1$, the space $(S^1)^{*r}$ is homeomorphic to S^{2r-1} . \square*

Remark 1. This homeomorphism can be realized as follows. Consider the space \mathbb{R}^{2r+1} of all real homogeneous polynomials $\mathbb{R}^2 \rightarrow \mathbb{R}^1$ of degree $2r$, the convex cone in this space consisting of everywhere non-negative polynomials, and (also convex) dual cone in the dual space $\widehat{\mathbb{R}}^{2r+1}$ consisting of linear forms taking only positive values inside the previous

cone. The intersection of the boundary of this dual cone with the unit sphere in $\widehat{\mathbb{R}}^{2r+1}$ is naturally homeomorphic to $(S^1)^{*r}$; on the other hand it is homeomorphic to the boundary of a convex $2r$ -dimensional domain.

Lemma 2 (see [8], Lemma 3). *For any $r > 1$ the group $H_*((S^2)^{*r}, \mathbb{C})$ is trivial in all positive dimensions.* \square

Consider the “sign local system” $\pm\mathbb{Q}$ over $B(\mathbb{CP}^1, p)$, i.e. the local system of groups with fiber \mathbb{Q} , such that the elements of $\pi_1(B(\mathbb{CP}^1, p))$ defining odd (respectively, even) permutations of p points in \mathbb{CP}^1 act in the fiber as multiplication by -1 (respectively, 1).

Lemma 3 (see [8], Lemma 2). *All groups $H_i(B(\mathbb{CP}^1, p); \pm\mathbb{Q})$ with $p \geq 1$ are trivial except only for $H_0(B(\mathbb{CP}^1, 1), \pm\mathbb{Q}) \sim H_2(B(\mathbb{CP}^1, 1), \pm\mathbb{Q}) \sim H_2(B(\mathbb{CP}^1, 2), \pm\mathbb{Q}) \sim \mathbb{Q}$.* \square

4. PROOF OF THEOREM 1

Following [1], we use the Alexander duality

$$(2) \quad \tilde{H}^i(\mathbb{R}^D \setminus \Sigma) \simeq \overline{H}_{D-i-1}(\Sigma),$$

where \overline{H}_* denotes the Borel—Moore homology.

4.1. Simplicial resolution of Σ . To calculate the right-hand group in (2) we construct a *resolution* of the space Σ . Let $\chi : \mathbb{RP}^1 \rightarrow \mathbb{R}^T$ be a generic embedding, $T \gg n$. For any system $\Phi = (f_1, \dots, f_n) \in \Sigma$, not equal identically to zero, consider the simplex $\Delta(\Phi)$ in \mathbb{R}^T , spanned by the images $\chi(x_i)$ of all points $x_i \in \mathbb{RP}^1$, corresponding to all possible lines, on which the system f has a common root. (The maximal possible number of such lines is obviously equal to d_1 .)

Further consider a subset in the direct product $\mathbb{R}^D \times \mathbb{R}^T$, namely the union of all simplices of the form $\Phi \times \Delta(\Phi)$, $\Phi \in \Sigma \setminus 0$. This union is not closed: the set of its limit points, not belonging to it, is the product of the point $0 \in \mathbb{R}^D$ (corresponding to the zero system) and the union of all simplices in \mathbb{R}^T , spanned by the images of no more than d_1 different points of the line \mathbb{RP}^1 . By the Caratheodory theorem, the latter union is homeomorphic to the sphere S^{2d_1-1} . We can assume that our embedding $\chi : \mathbb{RP}^1 \rightarrow \mathbb{R}^T$ is algebraic, and hence this sphere is semialgebraic. Take a generic $2d_1$ -dimensional semialgebraic disc in \mathbb{R}^T with boundary at this sphere (e.g., the union of segments connecting the points of this sphere with a generic point in \mathbb{R}^T) and add the product of the point $0 \in \mathbb{R}^D$ and this disc to the previous union of simplices in $\mathbb{R}^D \times \mathbb{R}^T$. The resulting set will be denoted by σ and called a *simplicial resolution* of Σ .

Lemma 4. *The obvious projection $\sigma \rightarrow \Sigma$ (induced by the projection of $\mathbb{R}^D \times \mathbb{R}^T$ onto the first factor) is proper, and the induced map of one-point compactifications of these spaces is a homotopy equivalence.*

This follows easily from the fact that this projection is a stratified map of semialgebraic spaces, and the preimage of any point $\bar{\Sigma}$ is contractible, cf. [5], [6]. \square

So, we can (and will) calculate the group $\overline{H}_*(\sigma)$ instead of $\overline{H}_*(\Sigma)$.

Remark 2. There is a more canonical construction of a simplicial resolution of Σ in the terms of “Hilbert schemes”. Namely, let I_p be the space of all ideals of codimension p in the space of smooth functions $\mathbb{R}^1 \rightarrow \mathbb{R}^1$, supplied with the natural “Grassmannian” topology. It is easy to see that I_p is homeomorphic to the p th symmetric power $S^p(\mathbb{R}^1) = (\mathbb{R}^1)^p / S(p)$, in particular it contains the configuration space $B(\mathbb{R}^1, p)$ as an open dense subset. Consider the disjoint union of these d_1 spaces I_1, \dots, I_{d_1} , augmented with the one-point set I_0 symbolizing the zero ideal. The incidence of ideals makes this union a partially ordered set. Consider the continuous order complex Ξ_{d_1} of this poset, i.e. the subset in the join $I_1 * \dots * I_{d_1} * I_0$ consisting of those simplices, all whose vertices are incident to one another. For any polynomial system $\Phi = (f_1, \dots, f_n) \in \mathbb{R}^D$ denote by $\Xi(\Phi)$ the subcomplex in Ξ_{d_1} consisting of all simplices, all whose vertices correspond to ideals containing all polynomials f_1, \dots, f_n . The simplicial resolution $\tilde{\sigma} \subset \Sigma \times \Xi_{d_1}$ is defined as the union of simplices $\Phi \times \Xi(\Phi)$ over all $\Phi \in \Sigma$.

This construction is homotopy equivalent to the previous one. In particular, the Caratheodory theorem has the following version: the continuous order complex of the poset of all ideals of codimension $\leq r$ in the space of functions $S^1 \rightarrow \mathbb{R}^1$ is *homotopy equivalent* to S^{2r-1} .

However, this construction is less convenient for our practical calculations.

The space σ has a natural increasing filtration $F_1 \subset \dots \subset F_{d_1+1} \equiv \sigma$: its term F_p , $p \leq d_1$, is the closure of the union of all simplices of the form $\Phi \times \Delta(\Phi)$ over all polynomial systems Φ having no more than p lines of common zeros.

Lemma 5. *For any $p = 1, \dots, d_1$, the term $F_p \setminus F_{p-1}$ of our filtration is the space of a locally trivial fiber bundle over the configuration space $B(\mathbb{R}^1, p)$, with fibers equal to the direct product of an $(p-1)$ -dimensional open simplex and an $(D - N(p))$ -dimensional real space. The corresponding bundle of open simplices is orientable if and only if*

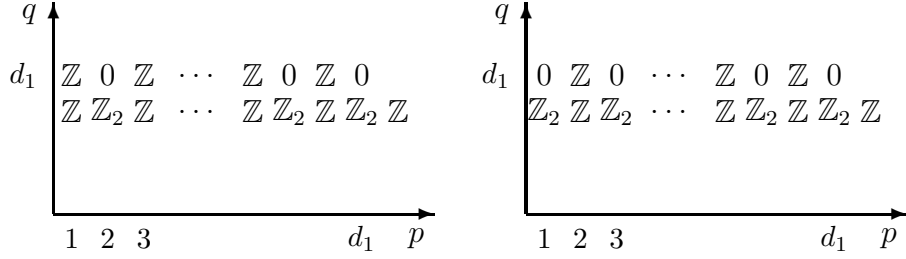


FIG. 1. E^1 for $n = 1$, d_1 even and $n = 1$, d_1 odd

p is odd (i.e. exactly when the base configuration space is orientable), and the bundle of $(D - N(p))$ -dimensional spaces is orientable if and only if the index $\Upsilon(p)$ is even.

The last term $F_{d_1+1} \setminus F_{d_1}$ of this filtration is homeomorphic to an open $2d_1$ -dimensional disc.

Indeed, to any configuration $(x_1, \dots, x_p) \in B(\mathbb{RP}^1, p)$, $p \leq d_1$, there corresponds the direct product of the interior part of the simplex in \mathbb{R}^T , spanned by the images $\chi(x_i)$ of points of this configuration, and the subspace in \mathbb{R}^D , consisting of polynomial systems, having solutions on corresponding p lines in \mathbb{R}^2 . The codimension of the latter subspace is equal exactly to $N(p)$. The assertion concerning the orientations can be checked elementary. The description of $F_{d_1+1} \setminus F_{d_1}$ follows immediately from the construction. \square

Consider the spectral sequence $E_{p,q}^r$, calculating the group $\overline{H}_*(\Sigma)$ and generated by this filtration. Its term $E_{p,q}^1$ is canonically isomorphic to the group $\overline{H}_{p+q}(F_p \setminus F_{p-1})$. By Lemma 5, its column $E_{p,*}^1$, $p \leq d_1$, is as follows. If $\Upsilon(p)$ is even, then it contains exactly two non-trivial terms $E_{p,q}^1$, both isomorphic to \mathbb{Z} , for q equal to $D - N(p) + p - 1$ and $D - N(p) + p - 2$. If $\Upsilon(p)$ is odd, then it contains only one non-trivial term $E_{p,q}^1$, isomorphic to \mathbb{Z}_2 , for $q = D - N(p) + p - 2$. Finally, the column $E_{d_1+1,*}^1$ contains only one non-trivial element $E_{d_1+1,d_1-1}^1 \sim \mathbb{Z}$.

Before calculating the differentials and further terms E^r , $r > 1$, let us consider several basic examples.

4.2. The case $n = 1$. If our system consists of only one polynomial of degree d_1 , then the term E^1 of our spectral sequence looks as in Fig. 1, in particular all non-trivial groups $E_{p,q}^1$ lie in two rows $q = d_1$ and $d_1 - 1$.

Lemma 6. *If $n = 1$ then in both cases of even or odd d_1 , all possible horizontal differentials $\partial_1 : E_{p,d_1-1}^1 \rightarrow E_{p-1,d_1-1}^1$ of the form $\mathbb{Z} \rightarrow \mathbb{Z}_2$,*

these groups die at the instant E^3 except for $E_{d_2+1, d_1-1}^3 \sim \mathbb{Z}$ for even $d_1 - d_2$, and $E_{d_2+2, d_1-1}^3 \sim \mathbb{Z}$ for odd $d_1 - d_2$.

In the case of even $d_1 - d_2$ all other differentials between the groups $E_{p,q}^r$ are trivial, because otherwise the group $\overline{H}^0(\mathbb{R}^d \setminus \Sigma)$ would be smaller than \mathbb{Z}^{d_2} , in contradiction to $d_2 + 1$ different components of this space indicated in Example 1.

On contrary, if $d_1 - d_2$ is odd, then all differentials $d_r : E_{d_2+2, d_1-1}^r \rightarrow E_{d_2+2-r, d_1-2+r}^r$, $r = 1, \dots, d_1 - d_2 + 1$ are epimorphic just because the integer cohomology group of the topological space $\mathbb{R}^D \setminus \Sigma$ cannot have non-trivial torsion subgroup in dimension 1. Therefore the unique non-trivial group $E_{p,q}^\infty$ in this case is $E_{d_2+2, d_1-1}^\infty \sim \mathbb{Z}$.

These considerations prove Main Theorem in the case $n = 2$.

4.4. The general case. Now suppose that our systems (1) consist of $n \geq 3$ polynomials. Let again σ be the simplicial resolution of the corresponding resultant variety, constructed in §4.1, and σ' be the simplicial resolution of the resultant variety for $n = 2$ and the same d_1 and d_2 . The parts $\sigma \setminus F_{d_3}(\sigma)$ and $\sigma' \setminus F_{d_3}(\sigma')$ of these resolutions are canonically homeomorphic to one another as filtered spaces. In particular, $E_{p,q}^1(\sigma) = E_{p,q}^1(\sigma')$ if $p > d_3$, and $E_{p,q}^r(\sigma) = E_{p,q}^r$ if $p \geq d_3 + r$. All non-trivial terms $E_{p,q}^r(\sigma)$ with $p \leq d_3$ are placed in such a way, that no non-trivial differentials ∂_r can act between these terms, as well as no differentials can act to these terms from the cells $E_{p,q}^r$ with $p > d_3$, which have survived the differentials between these cells, described in the previous subsection.

Therefore the final term $E_{p,q}^\infty(\sigma)$ coincides with $E_{p,q}^1(\sigma)$ in the domain $\{p \leq d_3\}$, and coincides with the term $E_{p,q}^\infty(\sigma')$ of the truncated spectral sequence calculating the Borel-Moore homology of $\sigma' \setminus F_{d_3}(\sigma')$ in the domain $\{p > d_3\}$. This terminates the proof of Theorem 1. \square

5. PROOF OF THEOREMS 2, 3

The simplicial resolution $\sigma_{\mathbb{C}}$ of $\Sigma_{\mathbb{C}}$ appears in the same way as its real analog σ in the previous section. It also has a natural filtration $F_1 \subset \dots \subset F_{d_1+1} \equiv \sigma_{\mathbb{C}}$. For $p \in [1, d_1]$ its term $F_p \setminus F_{p-1}$ is fibered over the configuration space $B(\mathbb{CP}^1, p)$; its fiber over a configuration (x_1, \dots, x_p) is equal to the product of the space $\mathbb{C}^{D-N(p)}$ (consisting of all complex systems (1) vanishing at all lines corresponding to the points of this configuration) and the $(p-1)$ -dimensional simplex, whose vertices correspond to the points of the configuration. In particular, our spectral sequence calculating rational Borel-Moore homology of $\sigma_{\mathbb{C}}$ has $E_{p,q}^1 \sim \overline{H}_{q-2(D-N(p))+1}(B(\mathbb{CP}^1, p); \pm\mathbb{Q})$ for such p . By Lemma

3, only the following such groups are non-trivial: $E_{1,2(D-n)-1}^1 \sim \mathbb{Q}$, $E_{1,2(D-n)+1}^1 \sim \mathbb{Q}$, and (if $d_1 > 1$) $E_{2,2(D-2n)+1}^1$.

The last term $F_{d_1+1} \setminus F_{d_1}$ is homeomorphic to the cone over the d_1 -th self-join $(\mathbb{CP}^1)^{*d_1}$ with the base of this cone removed (as it belongs to F_{d_1}). Therefore by Lemma 2 the column $E_{d_1+1,*}^1$ is trivial if $d_1 > 1$ and contains unique non-trivial group $E_{2,1}^1 \sim \mathbb{Q}$ if $d_1 = 1$.

So, in any case the first leaf E^1 of our spectral sequence has only three non-trivial terms $E_{1,2(D-n)-1}^1 \sim \mathbb{Q}$, $E_{1,2(D-n)+1}^1 \sim \mathbb{Q}$, and $E_{2,2(D-2n)+1}^1$. The differentials in it are obviously trivial, therefore the group $\overline{H}_*(\sigma)$ has three non-trivial terms in dimensions $2(D-n)$, $2(D-n)+2$, and $2(D-2n)+3$. By Alexander duality in the space \mathbb{C}^D this gives us three groups $\tilde{H}^{2n-3} \sim \mathbb{Q}$, $\tilde{H}^{2n-1} \sim \mathbb{Q}$ and $\tilde{H}^{4n-4} \sim \mathbb{Q}$ and zero in all other dimensions. \square

Theorem 3 can be proved in exactly the same way. \square

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